Bounds for the Torsional Rigidity of Heated Beams

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THE effective torsional rigidity of a beam, in the presence of an axial thermal stress σ_T, is well-known to be[1–5]

$$GC_{eff} = GC_0 + \int_A \sigma_T r^2 dA \tag{1}$$

where r is the distance from the center of twist, C_0 the torsion constant, A the area of the cross section, and G the shear modulus. In general, for a beam heated on the surface, $C_{eff} < C_0$, it will now be shown how some lower bounds to C_{eff} can be constructed.

The simplest such bound can be obtained if it is recalled that[6] the thermoelastic stress in a beam under arbitrary temperature distributions cannot exceed a value proportional to the maximum temperature difference ΔT within a cross section, or

$$|\sigma_T| \leq k\alpha E\Delta T; \quad T_M - \Delta T \leq T \leq T_M \tag{2}$$

where T_M is the maximum temperature and where the coefficient k, dependent only upon the cross-sectional shape, is easily calculated by means of the "column analogy" of Ref. 6. For example, $k = \frac{4}{3}$ for a solid rectangular beam, $k = \frac{2}{3} + 3^{1/2}/\pi$ for a thin-walled circular tube, and

$$k = \tfrac{1}{2} + [5 + 21\beta + 27\beta^2 + 9\beta^3]/[6(3\beta + 1) \times (\beta + 1)^2], \quad \beta = c/h \tag{3}$$

for the thin-walled rectangular tube referred to later as shape (C); furthermore, for a beam of any cross section under symmetric temperature distributions, k is always equal to unity. Values of k for composite beams were given in Refs. 7 and 8.

Substitution of Eq. (2) in Eq. (1) gives a lower bound to C_{eff} in the form

$$C_{eff}/C_0 \geq 1 - k\alpha EI_p\Delta T/GC_0 \tag{4}$$

where I_p is the polar moment of inertia about the center of twist. Alternatively, since, in general,[5] Eq. (1) can be written as

$$C_{eff}/C_0 = 1 - \Delta T/\Delta T_{cr} \tag{5}$$

where the critical temperature difference ΔT_{cr} is that for which $C_{eff} = 0$, a lower bound for ΔT_{cr} is easily seen by comparison of Eqs. (4) and (5) to be given by the relation

$$\Delta T_{cr} \geq GC_0/k\alpha EI_p \tag{6}$$

This limiting value for ΔT_{cr} may usually be expected to be somewhat too conservative because, in calculating it, the absolute maximum for σ_T was used at all points of the cross section. Some improvement could be obtained by employing the maximum σ_T possible at *each* point, although different temperature distributions are required to produce this stress of different points in the cross section.[6] The best results, however, are given by the particular temperature distribution which will make C_{eff} (or ΔT_{cr}) a minimum. To do this, and restricting for simplicity the analysis to the case of symmetry i.e., with

$$\sigma_T = -\alpha E\left(T - \frac{1}{A}\int_A TdA\right) \tag{7}$$

we start by rewriting Eq. (1) as

$$\frac{C_{eff}}{C_0} = 1 + \frac{\alpha E}{GC_0} \times \left[-\int_A Tr^2dA + \frac{1}{A}\int_A r^2dA \int_A TdA\right] = 1 + \frac{\alpha E}{GC_0}\int_A (\rho^2 - r^2)TdA \tag{8}$$

where $\rho = (I_p/A)^{1/2}$ is the radius of gyration of the section. Since the temperature varies only within the range defined in Eq. (2), clearly a lower bound for C_{eff} is obtained by using $T = T_M$, where $|r| > \rho$ and $T = T_M - \Delta T$ where $|r| < \rho$. Let A_0 and I_{p0} be, respectively, the area and the polar moment of inertia of the regions of the cross section for which $|r| > \rho$, with $(A - A_0)$ and $(I_p - I_{p0})$ the corresponding quantities for the remainder of the section. With this notation then

$$\frac{C_{eff}}{C_0} \geq 1 + \frac{\alpha E}{GC_0}\{T_M(A_0\rho^2 - I_{p0}) + (T_M - \Delta T) \times [\rho^2(A - A_0) - (I_p - I_{p0})]\} \tag{9}$$

Thus, the new bound is

$$C_{eff}/C_0 \geq 1 - [\alpha EI_p/GC_0][(I_{p0}/I_p) - A_0/A]\Delta T \tag{10a}$$

or, alternatively,

$$\Delta T_{cr} \geq GC_0/\{\alpha EI_p[(I_{p0}/I_p) - A_0/A]\} \tag{10b}$$

which is larger than that of Eq. (6), since it can be easily shown that the quantity $[(I_{p0}/I_p) - A_0/A]$ is smaller than unity.

The four thin doubly symmetrical cross-sectional shapes analyzed in Ref. 5 for chord-wise temperature distributions of the form

$$T = \Delta T(2x/c)^{2n}, \qquad |x| \leq c/2 \tag{11}$$

will be used here again for a discussion of the results. The four shapes are: (A) a solid rectangle, (B) a solid double wedge, (C) a rectangular box beam of wall thickness δ, and (D) a double-arc box beam of wall thickness δ. Each section has a maximum depth h and a chord c, and we assume that $h/c \ll 1$ and $\delta/h \ll 1$.

It may first be noted that the actual values of ΔT_{cr} (in Ref. 5), and both bounds calculated here are proportional to the quantity $GC_0/(\alpha EI_p)$, or, for the four thin sections in question, to the quantity

$$K = (G/\alpha E)(h/c)^2 \tag{12}$$

Thus, the correct dependence on the physical and geometrical parameters is predicted by either of the bounds; they may therefore be used to determine the *relative* merits of sections of different materials and/or aspect ratios. For example, if a single point is determined in plots (such as those of Figs. 8 and 12 of Ref. 3) of (GC_{eff}/GC_0) against (h/c) for various materials, the entire set of curves can immediately be constructed.

With $h/c = 0.04$ as an example, a numerical comparison can be set up as in Table 1.

Table 1 Comparison of exact and approximate bounds

Shape	A	B	C	D
Eq. (6)	4.0	4.0	8.6	4.7
Eq. (10b)	10.4	5.8	33.3	13.8
Ref. 5	14.9	15.0	36.3	20.0

The numbers listed are values of $\Delta T_{cr}/K$; those of the first row correspond to $k = 1$, since the temperature (Eq. 11) is

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symmetric, and those of the last row to the value of n for which Ref. 5 leads to a minimum (namely $n = 3^{1/2}/2$, 0.50, 0.90 and $3^{1/2}/2$ for the four sections, respectively). As expected, Eq. (6) gives too conservative values, and so does Eq. (10b) in the case of section B. For sections A, C, and D, the ratios of the values for the minimum ΔT_{cr} given in the last two rows are 0.70, 0.92, and 0.69, respectively; these may be considered sufficiently close to unity to be useful either for quick preliminary estimates or for rough checks on more accurate calculations. It may be indeed concluded that this will always be the case for all sections, except those in which the bulk of the material is concentrated, as in section B, away from the tip regions $|r| > \rho$.

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A Finite Element Solution for Saint-Venant Bending

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Nomenclature¶

A,a = area of cross section, area of element
$E_{(x,y)}$ = modulus of elasticity in the z direction
G_{xz},G_{yz} = torsional moduli on x and y axes of orthotropy
K_{1e},K_{2e} = curvature defined in Eq. (6)
P_x,P_y = end loads in the x and y direction applied at the shear center (Fig. 1)
$w_z,\bar{w},w$ = warping functions [Eq. (9) and (14)]
σ,ϵ = normal stress and strain
τ,γ = shearing stress and strain
ν_{zx},ν_{zy} = Poisson's ratio for strain in the x direction due to stress in the z direction, etc.

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¶ For definitions of properties of cross section such as A_e, M_{xe}, etc., see Eq. (5). See also basic nomenclature of Ref. 9.

1. Introduction

FOR moderately thick beams, the classical Euler-Bernoulli beam theory must be modified to include shear deformation, especially when such a beam is vibrating at frequencies above the fundamental. The equations which include this shear effect as well as rotatory inertia are due to Timoshenko.[1] In these equations, the shearing strain may be expressed as the average shearing stress divided by the shear modulus and a shear coefficient K. This dimensionless coefficient, which depends on the shape of the crosssection, must be introduced since neither the shearing stress nor the shearing strain are uniformly distributed over the crosssection.

A recent contribution[2] on evaluating an acceptable value of K requires an elasticity displacement solution for bending of beams subjected to end load. Saint-Venant solved this problem for a homogeneous isotropic beam. Recent solutions for isotropic and aeolotropic material, in terms of a stress function have been given by several authors.[3–5] The problem has also been solved for beams of nonhomogeneous material whose Poisson's ratio is constant.[6,7] Beams of only a few simple geometric sections have been solved exactly due to the mathematical complexity involved. Thus, an approximate solution is needed. A numerical solution will yield the transverse shear-stress distribution, the associated longitudinal displacements (warping), and the location of the shear center.

2. Saint-Venant Beam Theory for Nonhomogeneous Orthotropic Beams

The method of solution is to assume that the stresses σ_x, σ_y, and τ_{xy}, in the beam of Fig. 1 are zero and then show that, for either the isotropic or the orthotropic homogeneous beam, four of the compatibility equations can be satisfied if ϵ_z is of the form,

$$\epsilon_z = c_1zy + c_2zx + c_3x + c_4y + c_5z + c_6 \quad \sigma_z = E_{(x,y)}\epsilon_z \quad (1)$$

where c_1–c_6 are constants of integration. Equation (1) is also true for isotropic or orthotropic nonhomogeneous beams when all Poisson's ratios are constant. Boundary conditions to be satisfied are

$$\int_A \sigma_z dA = 0 \quad \int_A \sigma_z x dA = -P_x(l - z)$$
$$\int_A \sigma_z dA = -P_y(l - z) \quad (2)$$

If Eqs. (2) are satisfied then the boundary conditions

$$\int_A \tau_{xz} dA = P_x \quad \int_A \tau_{yz} dA = P_y \quad (3)$$

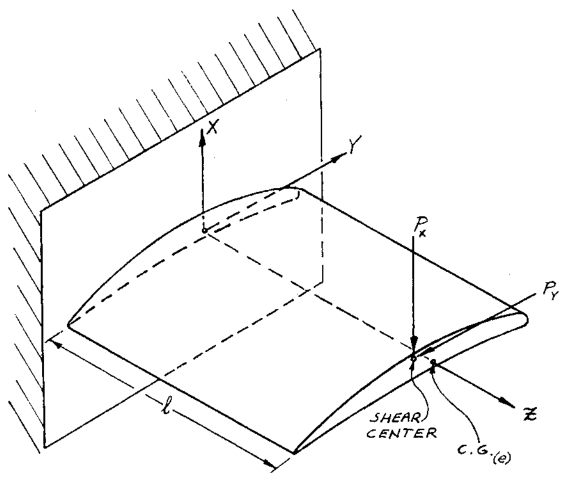

Fig. 1 Beam showing end loading.